

# HIGH DIMENSIONAL RANDOM SECTIONS OF ISOTROPIC CONVEX BODIES

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ABSTRACT. We study two properties of random high dimensional sections of convex bodies. In the first part of the paper we estimate the central section function  $|K \cap F^\perp|_{n-k}^{1/k}$  for random  $F \in G_{n,k}$  and  $K \subset \mathbb{R}^n$  a centrally symmetric isotropic convex body. This partially answers a question raised by V. Milman and A. Pajor (see [MP], p.88). In the second part we show that every symmetric convex body has random high dimensional sections  $F \in G_{n,k}$  with outer volume ratio bounded by

$$\text{ovr}(K \cap F) \leq C \frac{n}{n-k} \log \left( 1 + \frac{n}{n-k} \right)$$

## 1. INTRODUCTION AND NOTATION

Throughout the paper  $K \subset \mathbb{R}^n$  will denote a symmetric convex body.  $K$  is called isotropic if it is of volume 1 and its inertia matrix is a multiple of the identity. Equivalently, there exists a constant  $L_K > 0$  called isotropy constant of  $K$  such that  $L_K^2 = \int_K \langle x, \theta \rangle^2 dx, \forall \theta \in S^{n-1}$ .

The relation between the isotropy constant and the size of the central sections of an isotropic convex bodies appears in [H], [B] or [MP] where it is proved that for every  $1 \leq k \leq n$  there exist  $c_1(k), c_2(k) > 0$  such that for every subspace  $F \in G_{n,k}$  (the Grassmann space) and  $K \subset \mathbb{R}^n$  isotropic

$$\frac{c_1(k)}{L_K} \leq |K \cap F^\perp|_{n-k}^{1/k} \leq \frac{c_2(k)}{L_K}$$

where  $|\cdot|_m$  is the Lebesgue measure in the appropriate  $m$  dimensional space.

More precisely, it is proved in [MP] that  $|K \cap F^\perp|_{n-k}^{1/k} \sim L_{B_{k+1}(K,F)}/L_K$ , see Lemma 2.2 below for an explanation ( $a \sim b$  means  $a \cdot c_1 \leq b \leq a \cdot c_2$  for some numerical constants  $c_1, c_2 > 0$ ). From now on the letters  $c, C, c_1, \dots$  will denote absolute numerical constants, whose value may change from line to line. Well known estimates imply  $c_1(k) \geq c_1$ , [H], and  $c_2(k) \leq c_2 k^{1/4}$ , [K11]. We remark that these bounds are valid for *every* subspace  $F \in G_{n,k}$ .

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Our first main result of the paper is an improvement of this general result for “most” subspaces. Denoting by  $\mu$  the Haar probability on  $G_{n,k}$  we show

**Theorem 2.1** *There exist absolute constants  $c_1, c_2, c_3 > 0$  with the following property: If  $K$  is an isotropic convex body in  $\mathbb{R}^n$  and  $1 \leq k \leq \sqrt{n}$  then*

$$\mu\{F \in G_{n,k} : \frac{c_1}{L_K} \leq |K \cap F^\perp|_{n-k}^{1/k} \leq \frac{c_2}{L_K}\} \geq 1 - e^{-c_3 \frac{n}{k}}$$

The proof uses the tools developed in [Pa1].

Our first motivation for the second part of the paper was to give upper bounds for the isotropy constant  $L_{K \cap F}$  of high dimensional random sections of  $K$ . For that matter, we estimate the volume ratio (see definition below) of projections of convex bodies  $\text{vr}(P_E(K))$ . By a straightforward duality argument that is equivalent to having an estimate for the outer volume ratio of sections which is known to control their isotropy constant.

The notion of volume ratio was introduced by S. Szarek in the seventies, cf. [S], in the context of the local theory of normed spaces. As a consequence of John’s theorem, the volume ratio of  $K$  is bounded from above by  $\sqrt{n}$  and the bound is sharp up to a constant.

One of the main well known consequences of the  $M$ -position is the fact that random proportional projections of  $K$  (in  $M$ -position) have bounded volume ratio, see for instance [M2], [M3] or [P] Chapter 6, and references therein. In view of the applications, the stress is put in the key fact of being *proportional* and the general bound available is

$$\text{vr}(P_E(K)) \leq C^{\frac{n}{k} + \frac{n}{n-k}}$$

for random  $E \in G_{n,k}$  with probability greater than  $1 - e^{-n}$ . See also [K12] for a proof of this fact for non necessarily symmetric convex bodies in  $M$ -position. We should also mention [GM] on estimates of the diameter of random proportional sections, which also give upper bounds on the isotropy constant of proportional sections.

We will give estimates of the volume ratio  $\text{vr}(P_E(K))$  which yield non-trivial bounds even when the dimension  $k$  may be greater than proportional. More precisely, we will show

**Theorem 3.2** *For every symmetric convex body  $K$  in  $\mathbb{R}^n$  and any  $1 \leq k \leq n - 1$  there exists an  $F \in G_{n,k}$  such that*

$$\text{vr}(P_F(K)) \leq c \left( \frac{n}{n-k} \right)^{\frac{1}{2}} \log \left( 1 + \frac{n}{n-k} \right)$$

where  $c > 0$  is a universal constant.

The methods of random matrices appearing in [LPTJ] will enable us to prove the random version

**Theorem 3.3** *Let  $K$  be a centrally symmetric convex body in  $\mathbb{R}^n$ . For any  $k \geq \frac{n}{2}$  there exists a position of  $K$  such that the subset of  $G_{n,k}$  of subspaces*

$E$  satisfying

$$\text{vr}(P_E(K)) \leq C \frac{n}{n-k} \log \left( 1 + \frac{n}{n-k} \right)$$

has Haar probability larger than  $1 - 2e^{-\frac{n-k}{2}}$ .

In particular, our results imply the existence of one, respectively, random orthogonal sections of a convex body in  $M$ -position having the isotropy constant bounded by the bounds above, see Corollary 3.4.

We introduce some notation. We denote by  $|\cdot|$  the Euclidean norm in the appropriate space,  $D_n$  the Euclidean ball in  $\mathbb{R}^n$  and by  $\omega_n$  its Lebesgue measure. The surface area of the unit sphere is  $|S^{n-1}| = n \omega_n$ . For any  $k$ -dimensional subspace  $F \subset \mathbb{R}^n$  we denote  $S_F = S^{n-1} \cap F$ , the Haar probability on  $S_F$  by  $\sigma_F$ ,  $D_F = D_n \cap F$  and by  $P_F$  the orthogonal projection onto  $F$ . The Haar probability on the grassmaniann  $G_{n,k}$  is denoted by  $\mu$ .  $K^\circ = \{x \in \mathbb{R}^n; \langle x, y \rangle \leq 1\}$  denotes the polar body of  $K$ . For any convex body  $L \subset \mathbb{R}^n$  we will write  $\tilde{L} = L/|L|_n^{1/n}$ . We will denote  $W(K) := \int_{S^{n-1}} h_K(\theta) d\sigma(\theta)$ , the mean width of the convex body  $K$ .

## 2. IMPROVEMENT OF THE CONSTANTS VIA RANDOM SECTIONS

**Theorem 2.1.** *There exist absolute constants  $c_1, c_2, c_3 > 0$  with the following property: If  $K$  is an isotropic convex body in  $\mathbb{R}^n$  and  $1 \leq k \leq \sqrt{n}$  then*

$$(2.1) \quad \mu\{F \in G_{n,k} : \frac{c_1}{L_K} \leq |K \cap F^\perp|_{n-k}^{1/k} \leq \frac{c_2}{L_K}\} \geq 1 - e^{-c_3 \frac{n}{k}}$$

Through the section  $K$  will be (a symmetric convex body) of volume 1.

Let  $F$  be a  $k$ -dimensional subspace of  $\mathbb{R}^n$  and denote by  $E$  the orthogonal subspace of  $F$ . For every  $\phi \in S_F$  we define  $E(\phi) = \text{span}\{E, \phi\}$ .

K. Ball (see [B]) proved the following theorem: For every  $q \geq 0$  and  $\phi \in F$ , the function

$$\phi \mapsto |\phi|^{1+\frac{q}{q+1}} \left( \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^q dx \right)^{-\frac{1}{q+1}}$$

is a norm on  $F$ . We denote by  $B_q(K, F)$  the unit ball of this norm.

Under this notation it was proved in [MP] the following

**Lemma 2.2.** *If  $K$  is isotropic then  $B_{k+1}(K, F)$  is also isotropic for every  $F \in G_{n,k}$ , and*

$$(2.2) \quad |K \cap F^\perp|_{n-k}^{1/k} \sim \frac{L_{B_{k+1}(K, F)}}{L_K} \quad \forall F \in G_{n,k}$$

A generalization for  $L_q$  centroid bodies of this approach appeared in [Pa1]. For any  $q \geq 1$  we define the  $L_q$  centroid body of  $K$ , the symmetric convex body that has support function

$$h_{Z_q(K)}(z) := \left( \int_K |\langle x, z \rangle|^q dx \right)^{1/q}, \quad \forall z \in S^{n-1}$$

The following equality was proved in [Pa1]: For every  $1 \leq k \leq n-1$ ,  $F \in G_{n,k}$  and  $q \geq 1$ ,

$$(2.3) \quad P_F(Z_q(K)) = \left(\frac{k+q}{2}\right)^{1/q} |B_{k+q-1}(K, F)|_k^{1/k+1/q} Z_q(\tilde{B}_{k+q-1}(K, F))$$

**Proposition 2.3.** *Let  $K \subset \mathbb{R}^n$ ,  $1 \leq k \leq n-1$ ,  $F \in G_{n,k}$  and  $E = F^\perp$ . Then*

$$c_1 \leq |P_F Z_k(K)|_k^{1/k} |K \cap E|_{n-k}^{1/k} \leq c_2$$

where  $c_1, c_2 > 0$  are universal constants.

*Proof.* We choose  $q = k$  in (2.3). Then by taking volumes we have that

$$|P_F(Z_k(K))|_k^{1/k} = k^{1/k} |B_{2k-1}(K, F)|_k^{2/k} |Z_k(\tilde{B}_{2k-1}(K, F))|_k^{1/k}$$

It is known that there exists a universal constant  $c > 0$  such that for any symmetric convex body  $K$  of volume 1 in  $\mathbb{R}^k$  and  $q \geq k$ ,  $cK \subseteq Z_q(K) \subseteq K$ . (see [Pa2]) So,

$$c \leq |Z_k(\tilde{B}_{2k-1}(K, F))|_k^{1/k} \leq 1.$$

So, it is enough to prove that there exists  $c > 0$  such that

$$(2.4) \quad \frac{1}{|K \cap E|_{n-k}^{1/k}} \leq k^{1/k} |B_{2k-1}(K, F)|_k^{2/k} \leq c \frac{1}{|K \cap E|_{n-k}^{1/k}}$$

The right hand side inequality was proved in [Pa1]. The left hand side inequality follows the same line. We will need the following fact (see [MP] for a proof):

*Let  $C$  be a symmetric convex body in  $\mathbb{R}^m$ . If  $s \leq r$  are non-negative integers and  $\theta \in S^{m-1}$ , we have that*

$$(2.5) \quad \left(\frac{r+1}{2} \frac{\int_C |\langle x, \theta \rangle|^r dx}{|C \cap \theta^\perp|_{m-1}}\right)^{1/(r+1)} \geq \left(\frac{s+1}{2} \frac{\int_C |\langle x, \theta \rangle|^s dx}{|C \cap \theta^\perp|_{m-1}}\right)^{1/(s+1)}$$

Writing in polar coordinates we get

$$(2.6) \quad |B_{2k-1}(K, F)|_k = \omega_k \int_{S_F} \left( \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{2k-1} dx \right)^{\frac{1}{2}} d\sigma_F(\phi)$$

Applying (2.5) with  $C = K \cap E(\phi)$ ,  $m = n - k + 1$ ,  $r = 2k - 1$  and  $s = k - 1$ , we get

$$\left( k \frac{\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{2k-1} dx}{|K \cap E|_{n-k}} \right)^{1/(2k)} \geq \left( \frac{k}{2} \frac{\int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx}{|K \cap E|_{n-k}} \right)^{1/k}$$

It follows that

$$\left( \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{2k-1} dx \right)^{\frac{1}{2}} \geq (k |K \cap E|_{n-k})^{-1/2} \frac{k}{2} \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx$$

Then formula (2.6) becomes

$$|B_{2k-1}(K, F)|_k \geq (k|K \cap E|_{n-k})^{-1/2} \frac{k}{2} \omega_k \int_{S_F} \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx d\sigma_F(\phi)$$

Observe that (see also [Pa1])

$$|K|_n = \frac{k\omega_k}{2} \int_{S_F} \int_{K \cap E(\phi)} |\langle x, \phi \rangle|^{k-1} dx d\sigma_F(\phi).$$

So we get  $|B_{2k-1}(K, F)|_k \geq \frac{1}{k^{1/2}} |K \cap E|_{n-k}^{-1/2}$ , that is

$$k^{1/k} |B_{2k-1}(K, F)|_k^{2/k} \geq \frac{1}{|K \cap E|_{n-k}^{1/k}}$$

That proves formula (2.4) and the Proposition.  $\square$

We will use the isomorphic version of Dvoretzky theorem proved by V. Milman (see [M1], [MS]):

**Proposition 2.4.** *Let  $C$  a symmetric convex body in  $\mathbb{R}^n$ . If  $k \leq c_1 n \left(\frac{W(C)}{R(C)}\right)^2$*

$$(2.7) \quad \mu\{F \in G_{n,k} : \frac{W(C)}{2} D_F \subseteq P_F(C) \subseteq 2W(C) D_F\} \geq 1 - \exp -c_2 n \left(\frac{W(C)}{R(C)}\right)^2$$

where  $c_1, c_2 > 0$  are universal constants,  $R(C) = \max\{|x| ; x \in C\}$  and  $W(C) = \int_{S^{n-1}} h_C(\theta) d\sigma(\theta)$

We will denote  $k_*(C^\circ) := n \left(\frac{W(C)}{R(C)}\right)^2$ . Furthermore, (see [LMS]) we have that for  $p \leq k_*(C^\circ)$ ,

$$(2.8) \quad W(C) \sim W_p(C) := \left( \int_{S^{n-1}} h_C^p(\theta) d\sigma(\theta) \right)^{1/p}$$

**Definition 2.5.** Let  $K$  be a symmetric convex body of volume 1 in  $\mathbb{R}^n$  and let  $\alpha \in [1, 2]$ . We say that  $K$  is a  $\psi_\alpha$ -body with constant  $b_\alpha$  if

$$\left( \int_K |\langle x, \theta \rangle|^q dx \right)^{1/q} \leq b_\alpha q^{1/\alpha} \left( \int_K |\langle x, \theta \rangle|^2 dx \right)^{1/2}$$

for all  $q \geq \alpha$  and all  $\theta \in S^{n-1}$ . Equivalently, if

$$Z_q(K) \subseteq b_\alpha q^{1/\alpha} Z_2(K) \quad \text{for all } q \geq \alpha$$

The following definition appeared in [Pa3]:

**Definition 2.6.** Let  $K$  be a symmetric convex body of volume 1 in  $\mathbb{R}^n$ . We define

$$q_*(K) = \max\{q \in \mathbb{N} : k_*(Z_q^\circ(K)) \geq q\}$$

where  $Z_q^\circ(K) := (Z_q(K))^\circ$ .

We will need the following lower bounds for the quantities  $k_*(Z_q^\circ(K))$  and  $q_*(K)$ , (see [Pa1]):

**Proposition 2.7.** *Let  $K$  be an isotropic  $\psi_\alpha$ -body with constant  $b_\alpha$  and  $1 \leq q \leq n$  then*

$$k_*(Z_q^\circ(K)) \geq c_1 \frac{n}{q^{\frac{2-\alpha}{\alpha}} b_\alpha^2} \quad \text{and} \quad q_*(K) \geq c_2 \left( \frac{\sqrt{n}}{b_\alpha} \right)^\alpha$$

**Proposition 2.8.** *Let  $K \subset \mathbb{R}^n$  be isotropic. Then for  $q \leq q_*(K)$  we have*

$$(2.9) \quad W(Z_q(K)) \sim \sqrt{q} L_K$$

*Proof.* A direct computation shows that for  $q \leq n$ ,

$$\left( \int_K |x|^q dx \right)^{1/q} \sim \sqrt{\frac{n}{q}} W_q(Z_q(K))$$

It was proved in [Pa1] that for every  $q \leq q_*(K)$  we have

$$\left( \int_K |x|^q dx \right)^{1/q} \sim \sqrt{n} L_K$$

Also by (2.8) and the definition of  $q_*(K)$  we have that

$$W(Z_q(K)) \sim W_q(Z_q(K)) \quad \forall q \leq q_*(K)$$

By putting these results together we conclude the proof.  $\square$

A well known application of Brunn-Minkowski inequality implies that every convex body is  $\psi_1$  body with a constant  $c$ , where  $c$  is universal. So, Theorem 2.1 is a direct consequence of the following:

**Theorem 2.9.** *Let  $K$  be an isotropic  $\psi_\alpha$ -body with constant  $b_\alpha$  and  $1 \leq k \leq c \left( \frac{\sqrt{n}}{b_\alpha} \right)^\alpha$ . Then*

$$\mu\{F \in G_{n,k} : \frac{c_1}{L_K} \leq |K \cap F^\perp|_{n-k}^{1/k} \leq \frac{c_2}{L_K}\} \geq 1 - \exp\left\{-c_3 \frac{n}{k^{\frac{2-\alpha}{\alpha}} b_\alpha^2}\right\}$$

*Proof.* Let  $1 \leq k \leq q_*(K)$ . We will apply Proposition 2.4 for the symmetric convex body  $Z_k(K)$ . So, we have that there exists an  $A \subseteq G_{n,k}$  of measure greater than  $1 - e^{-c_2 k_*(Z_k^\circ(K))}$  such that for every  $F \in A$

$$\frac{W(Z_k(K))}{2} D_F \subseteq P_F(Z_k(K)) \subseteq 2W(Z_k(K)) D_F$$

By taking volumes we get

$$|P_F(Z_k(K))|_k^{1/k} \sim \frac{W(Z_k(K))}{\sqrt{k}} \sim L_K$$

where we used Proposition 2.8 and the fact that  $|D_k|_k^{1/k} \sim \frac{1}{\sqrt{k}}$ .

By Proposition 2.3 we get that for every  $F \in A$ ,

$$\frac{c_1}{L_K} \leq |K \cap E|_{n-k}^{1/k} \leq \frac{c_2}{L_K}$$

The result follows by Proposition 2.7.

□

*Remark 2.10.* By using a very recent version of the central limit theorem for convex bodies by R. Eldan and B. Klartag, [EK] one can deduce:

Let  $K$  be an isotropic convex body in  $\mathbb{R}^n$  and  $1 \leq k \leq n^c$ . Then there exists a subset  $\mathcal{E} \subset G_{n,k}$  with  $\mu(\mathcal{E}) \geq 1 - C \exp(-n^c)$  such that for any  $F \in \mathcal{E}$  we have

$$\frac{1 - \varepsilon}{\sqrt{2\pi}L_K} \leq |K \cap F^\perp|_{n-k}^{1/k} \leq \frac{1 + \varepsilon}{\sqrt{2\pi}L_K}$$

whenever  $0 < \varepsilon \sim \frac{1}{n^{c'}}.$

This result is different from Theorem 2.1. It produces optimal constants  $c_1(k), c_2(k)$ , but Theorem 2.1 is valid for  $k \leq \sqrt{n}$  (while  $c$  is very small).

For smaller values of  $k$ , namely  $1 \leq k \leq \frac{\varepsilon \log n}{(\log \log n)^2}$ , a simplified proof of the R. Eldan and B. Klartag theorem can be given. Indeed, it is possible estimate the Lipschitz constant of the section function  $G_{n,k} \ni E \rightarrow |E^\perp \cap K|_{n-k}$  and use the concentration of measure phenomenon on the Grassmannian to measure the closeness of this function with respect to its mean. Then, by expressing this expectation as a marginal, one relates it to the marginal of the gaussian distribution and use the concentration of  $|\cdot|$  on  $K$  in the version stated in [BB] (Theorems 3.5 and Corollary 3.6). We omit the details.

### 3. THE ISOTROPY CONSTANT OF HIGH DIMENSIONAL SECTIONS

In this section we give upper bounds for the isotropy constant of high dimensional sections of convex bodies by giving upper bounds for the volume ratio of high dimensional projections. First of all let us recall some very well known facts.

**Definition 3.1.** Let  $K \subseteq \mathbb{R}^n$  be a convex body. The volume ratio and outer volume ratio of  $K$  are defined as

$$\text{vr}(K) = \inf \left( \frac{|K|}{|\mathcal{E}|} \right)^{\frac{1}{n}} \quad \text{and} \quad \text{ovr}(K) = \inf \left( \frac{|\mathcal{E}'|}{|K|} \right)^{\frac{1}{n}}$$

where  $\mathcal{E}$  runs over all the ellipsoids contained in  $K$  and  $\mathcal{E}'$  runs over all the ellipsoids containing  $K$ .

The volume ratio of any centrally symmetric convex body is equivalent to the outer volume ratio of its dual, since if  $\mathcal{E}$  is the maximum volume ellipsoid contained in  $K$ , then

$$\frac{\text{vr}(K)}{\text{ovr}(K^\circ)} = \left( \frac{|K||K^\circ|}{|\mathcal{E}||\mathcal{E}^\circ|} \right)^{\frac{1}{n}}$$

and by Blaschke-Santaló inequality and its reverse there exists an absolute constant  $c$  such that

$$c \text{ovr}(K^\circ) \leq \text{vr}(K) \leq \text{ovr}(K^\circ).$$

Recall the identity  $K^\circ \cap F = (P_F(K))^\circ$ ,  $F \in G_{n,k}$ . Thus, by the previous formula we have

$$\text{ovr}(K^\circ \cap F) \sim \text{vr}(P_F(K))$$

It is also easy to prove that outer volume ratio bounds the isotropy constant of any convex body. The argument is,

$$nL_K^2 \leq \inf_{TK \subseteq D_n} \frac{1}{|TK|^{\frac{2}{n}}} \int_K |Ty|^2 \frac{dy}{|K|} \leq \inf_{TK \subseteq D_n} \frac{|\det T|^{\frac{-2}{n}} |D_n|^{\frac{2}{n}}}{|K|^{\frac{2}{n}} |D_n|^{\frac{2}{n}}} \leq Cn \text{ovr}(K)^2$$

The last ingredient we need is the existence of an  $M$ -position associated to a centrally symmetric convex body as it appears, for instance, in [P], Chapter 7:

Let  $0 < p < 2$ . There exists a linear map  $u: \ell_2^n \rightarrow (\mathbb{R}^n, \|\cdot\|_K)$  and universal constants  $c, c' > 0$  such that for all  $1 \leq k \leq n$ ,

- $c_k(u^{-1}), c_k(u^{-1*}) \leq \frac{c}{\sqrt{2-p}} \left(\frac{n}{k}\right)^{1/p}$  and
- $e_k(u), e_k(u^*) \leq \frac{c'}{\sqrt{2-p}} \left(\frac{n}{k}\right)^{1/p}$

where  $u^*$  is the adjoint operator,  $c_k$  are the Gelfand numbers and  $e_k$  the entropy numbers, defined by  $c_k(u^{-1}) = \inf\{\|u^{-1}|_S\|; \text{codim}(S) < k\}$  and  $e_k(u) = \inf\{t > 0; N(u(D_n), tK) \leq 2^{k-1}\}$  where

$$N(K, L) = \inf\{N \in \mathbb{N} \mid \exists x_1, \dots, x_N, K \subset \cup_{i=1}^N (x_i + L)\}$$

is the covering number of  $K$  by  $L$ .

**Theorem 3.2.** For every symmetric convex body  $K$  in  $\mathbb{R}^n$  and any  $1 \leq k \leq n-1$  there exists an  $F \in G_{n,k}$  such that

$$\text{vr}(P_F(K)) \leq c \left(\frac{n}{n-k}\right)^{\frac{1}{2}} \log\left(1 + \frac{n}{n-k}\right)$$

where  $c > 0$  is a universal constant.

*Proof.* Without loss of generality we can assume that  $|K| = 1$ . We use Pisier's result for fixed  $0 < p < 2$  and  $K^\circ$  and denote  $K_1 := u^{-1}(K^\circ)$ , that is, we take  $K^\circ$  into the so called  $M$ -position.

By definition of the entropy number we have  $N(u(D_n), e_k(u)K^\circ) \leq 2^{k-1}$  and so

$$(3.10) \quad D_n \subseteq \bigcup_{i=1}^{2^{k-1}} x_i + e_k(u)K_1.$$

By definition of Gelfand numbers, there exists a subspace  $S$  of codimension  $< k$  i.e. dimension  $> n - k$ , such that

$$|u^{-1}(x)| \leq \frac{c}{\sqrt{2-p}} \left(\frac{n}{k}\right)^{1/p} \quad \forall x \in K^\circ \cap S$$



and so it follows (by the definition of  $K_1$ ) that for every  $1 \leq k \leq n$ , there exists a subspace  $F \in G_{n,k}$  such that

$$(3.11) \quad R(K_1 \cap F) \leq \frac{c}{\sqrt{2-p}} \left( \frac{n}{n-k} \right)^{\frac{1}{p}}$$

$$(3.12) \quad K_1 \cap F \subseteq \frac{c}{\sqrt{2-p}} \left( \frac{n}{n-k} \right)^{\frac{1}{p}} D_F$$

From (3.10), we have that

$$D_F = D_n \cap F \subseteq \bigcup_{i=1}^{2^{k-1}} ((x_i + e_k(u)K_1) \cap F)$$

so

$$|D_F| \leq \sum_{i=1}^{2^{k-1}} |((x_i + e_k(u)K_1) \cap F)| \leq 2^{k-1} |e_k(u)K_1 \cap F|$$

Hence

$$\frac{1}{|K_1 \cap F|^{\frac{1}{k}}} \leq c |D_F|^{-\frac{1}{k}} e_k(u) \leq \frac{c' \sqrt{k}}{\sqrt{2-p}} \left( \frac{n}{k} \right)^{1/p}$$

Then there exists a subspace  $F \in G_{n,k}$  such that

$$\text{ovr}(K_1 \cap F) \leq \frac{c}{(2-p)} \left( \frac{n}{n-k} \right)^{\frac{1}{p}} \left( \frac{n}{k} \right)^{\frac{1}{p}}$$

Assume  $n/2 \leq k \leq n-1$ , i.e.  $1 \leq \frac{n}{k} \leq 2$  and  $2 \leq \frac{n}{n-k} \leq n$ . Thus, taking  $p = 2 - \frac{1}{\log(1 + \frac{n}{n-k})}$  we obtain that there exists  $F \in G_{n,k}$  such that

$$\text{ovr}(K_1 \cap F) \leq c \log \left( 1 + \frac{n}{n-k} \right) \left( \frac{n}{n-k} \right)^{\frac{1}{p}} \leq c \log \left( 1 + \frac{n}{n-k} \right) \left( \frac{n}{n-k} \right)^{\frac{1}{2}}$$

For any  $T \in GL(n)$  we have  $TK^\circ \cap F = T(K^\circ \cap T^{-1}F)$ . So we have proved that for every symmetric convex body  $K$  in  $\mathbb{R}^n$  and for any  $\frac{n}{2} \leq k \leq n-1$  there exists a subspace  $F$  of dimension  $k$  such that

$$\text{ovr}(K^\circ \cap F) \leq c \log \left( 1 + \frac{n}{n-k} \right) \left( \frac{n}{n-k} \right)^{\frac{1}{2}}$$

Observe that for  $k = \frac{n}{2}$ , the latter formula says that there exists  $F \in G_{n, \frac{n}{2}}$  and a constant  $C$  such that

$$\text{ovr}(K^\circ \cap F) \leq C$$

If  $1 \leq k \leq \frac{n}{2}$ , take  $E \in G_{n, 2k}$  and write  $K_0 = K^\circ \cap E$ . By the previous observation there exists a subspace  $F \subset E$  of dimension  $k$  such that  $\text{ovr}(K^\circ \cap F) = \text{ovr}(K_0 \cap F) \leq C$  and hence, since in this case  $\log(1 + \frac{n}{n-k}) \left( \frac{n}{n-k} \right)^{\frac{1}{2}}$  is between two absolute constants, the result also holds.  $\square$

The previous result gives the existence of one  $k$ -dimensional subspace  $F$  with the volume ratio of the projection on  $F$  bounded by a certain quantity. If  $k \geq \frac{n}{2}$ , we can enlarge the existence to random projections (in  $M$ -position) but we pay a penalty in the estimate of the volume ratio. The factor  $\frac{1}{2}$  in the hypothesis is irrelevant as long as we assume  $k \geq cn$ . The result is the following:

**Theorem 3.3.** *Let  $K$  be a centrally symmetric convex body in  $\mathbb{R}^n$ . For any  $k \geq \frac{n}{2}$  there exists a position of  $K$  such that the subset of  $G_{n,k}$  of subspaces  $E$  satisfying*

$$\text{vr}(P_E(K)) \leq C \frac{n}{n-k} \log \left( 1 + \frac{n}{n-k} \right)$$

has Haar probability larger than  $1 - 2e^{-\frac{n-k}{2}}$

*Proof.* Let  $K \subseteq \mathbb{R}^n$  be a symmetric convex body and let  $0 < p < 2$ . We take  $K_1 = u^{-1}(K^\circ)$  in  $M$ -position as in the previous Theorem. Moreover, as we saw above, for any  $k$ -dimensional subspace

$$\frac{1}{|K_1 \cap E|^{\frac{1}{k}}} \leq \frac{c' \sqrt{k}}{\sqrt{2-p}} \left( \frac{n}{k} \right)^{1/p}$$

In order to give an estimate for the diameter of a random section we use the random matrices method. Let  $G = (g_{i,j})$  a  $(n-k \times n)$  Gaussian random matrix, i.e. the  $g_{i,j}$ 's are i.i.d.  $N(0, 1/n)$  Gaussian entries (remark that  $g_{i,j} = \frac{1}{\sqrt{n}}g$  in law where  $g$  is a  $N(0, 1)$  Gaussian). Since  $G$  is rotationally invariant it induces the Haar measure in the grassmannian  $G_{n,k}$ , i.e. for any borelian  $A \subseteq G_{n,k}$

$$\mathbb{P}\{\text{Ker } G \in A\} = \mu\{E \in G_{n,k}; E \in A\}$$

The following fact is well known [DS], Th 2.13.

$$\mathbb{P}\left\{\|G : \ell_2^n \rightarrow \ell_2^{n-k}\| < 3\right\} \geq 1 - e^{-\frac{n-k}{2}} \quad (1)$$

Also if  $|x_0| = 1$  and  $0 < \eta$ ,

$$\mathbb{P}\{|Gx_0| < \eta\} = \int_{|x| \leq \eta\sqrt{n}} e^{-|x|^2/2} \frac{dx}{(\sqrt{2\pi})^{n-k}} \leq \left(C\eta\sqrt{\frac{n}{n-k}}\right)^{n-k} \quad (2)$$

for some absolute constant  $C > 0$ .

We follow [LPTJ] Proposition 2.3. Under same hypothesis as in preceding proposition we know that the entropy numbers  $e_{n-k}(u^{-1}) \leq \frac{C}{\sqrt{2-p}} \left(\frac{n}{n-k}\right)^{\frac{1}{p}}$  for  $1 \leq k \leq n$ . In particular it means that

$$K_1 \subset \bigcup_{j=1}^{2^{n-k}} x_j + \frac{2C}{\sqrt{2-p}} \left(\frac{n}{n-k}\right)^{1/p} D_n$$

Assume that  $G$  verifies (1) and the corresponding  $x_j$  verify (2) for  $\eta = C_1 \sqrt{\frac{n-k}{n}}$  ( $C_1$  to be chosen later). Let  $x \in K_1 \cap \text{Ker } G$ . Then for some  $x_j$ ,

$$\begin{aligned} |x| &\leq |x - x_j| + |x_j| < |x - x_j| + \frac{|Gx_j|}{\eta} = |x - x_j| + \frac{|G(x - x_j)|}{\eta} \\ &\leq |x - x_j| \left(1 + \frac{\|G\|}{\eta}\right) \leq \frac{C}{\sqrt{2-p}} \left(\frac{n}{n-k}\right)^{1/p+1/2} \end{aligned}$$

The preceding estimate is true with probability greater or equal than

$$1 - e^{-(n-k)/2} - 2^{(n-k)} C_1^{n-k} \geq 1 - 2e^{-(n-k)/2}$$

whenever  $2C_1 \leq \frac{1}{\sqrt{e}}$ .

We have proved that, with probability larger than  $1 - 2e^{-\frac{n-k}{2}}$ ,

$$R(K_1 \cap E) \leq \frac{C}{C_1 \sqrt{2-p}} \left(\frac{n}{n-k}\right)^{1/p+1/2}$$

Hence, as before,

$$\text{ovr}(K_1 \cap E) \leq \frac{c \left(\frac{n}{k}\right)^{1/p}}{(2-p)} \left(\frac{n}{n-k}\right)^{\frac{1}{p}+\frac{1}{2}}$$

with probability larger than  $1 - 2e^{-\frac{n-k}{2}}$ . Now, if we take  $p = 2 - \frac{1}{\log(1 + \frac{n}{n-k})}$  we have

$$\text{ovr}(K_1 \cap E) \leq c \left(\frac{n}{k}\right)^{\frac{1}{2 - \frac{1}{\log(1 + \frac{n}{n-k})}}} \frac{n}{n-k} \log \left(1 + \frac{n}{n-k}\right)$$

with probability larger than  $1 - 2e^{-\frac{n-k}{2}}$ . Since  $k \geq \frac{n}{2}$  we obtain the result.  $\square$

Since both  $K$  and  $K^\circ$  are simultaneously in  $M$ -position, by the remarks in the beginning of the section we readily obtain the following

**Corollary 3.4.** *Let  $K$  be a centrally symmetric convex body in  $\mathbb{R}^n$ . For every  $1 \leq k \leq n-1$  there exists an  $E \in G_{n,k}$  such that*

$$L_{K \cap E} \leq c \sqrt{\frac{n}{n-k}} \log \left(1 + \frac{n}{n-k}\right)$$

where  $c > 0$  is a universal constant. Moreover for any  $k \geq \frac{n}{2}$  there exists a position of  $K$  such that the subset of  $G_{n,k}$  of subspaces  $E$  satisfying simultaneously

$$\begin{aligned} L_{K \cap E} &\leq C \frac{n}{n-k} \log \left(1 + \frac{n}{n-k}\right) \\ L_{K^\circ \cap E} &\leq c \frac{n}{n-k} \log \left(1 + \frac{n}{n-k}\right) \end{aligned}$$

has Haar probability larger than  $1 - ce^{-\frac{n-k}{2}}$ .

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