Monografías de la Real Academia de Ciencias de Zaragoza. **26:** 89–104, (2004).

The Classification of the Finite Simple Groups: An Overview *

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Abstract

We briefly survey on the classification of finite simple groups.

A.M.S. Classification: 20D05, 20D06, 20D08, 20-02.

1 Introduction

The Theory of Groups is one of the oldest and most established branches of Algebra. Nowadays, the concept of group appears not only in the Theory of Groups itself, Algebra and other areas of Mathematics but in different parts of Science, as Physics, Chemistry, Chrystallography, and even Arts. In these areas, the concept of group arises as a kind of measure of the symmetry of a certain configuration to give deep information about it (see [31, 52]).

As it is well-known, different numerical systems form group under a certain law. For example, integers under sum, rationals under sum, non-zero rationals under multiplication, and so on. This is not the right origin of the concept of group. Rather we should take the point of view that this concept is the abstraction of common ideas of some areas (see [37]) as the following ones:

(1) Geometry at the beginning of the 19^{th} Century (*Erlangen Programm*)

(2) Number Theory at the end of 18^{th} Century (*Modular Arithmetic*)

(3) Algebraic Equations at the end of the 18^{th} Century (*Permutations*)

linked to important mathematicians as *Klein, Gauss, Lagrange, Ruffini, Cauchy* and many others. Special efforts are within the work of Galois in the Theory of Algebraic Equations,

*This article is based on a lecture given at a conference held at the Faculty of Sciences of Zaragoza (Spain) in 2004, organized by the Faculty and the Royal Academy of Sciences of Zaragoza

†This research was supported by Proyecto BFM2001-2452 of CICYT (Spain) and Proyecto 100/2001 of Gobierno de Aragón (Spain)

now known as *Galois Theory* (see [30, 46]), as from this work the ideas of "group of an algebraic equation", "normal subgroup", ... are introduced. After some additional work on groups as *permutation groups* (i.e. *Sylow* [45]), the abstract concept of group appears as a major contribution of *Cayley, von Dyck, Kronecker,* and specially *Burnside* (see [38]). In fact, the Theory of Groups came of age with the book by Burnside *Theory of groups of finite order* published in 1897 (see [11]).

1.1 Definitions

The modern definition of a group is usually given in the following way (see [18, 32, 42]). *A group* G is a non-empty set endowed with a binary operation $G \times G \longrightarrow G$ which assigns to every ordered pair of elements $x, y \in G$ a unique element of G (called *the product of* x *and* y) denoted by xy satisfying the following properties:

(1) *Associative law*: if $x, y, z \in G$ then $x(yz) = (xy)z$.
(2) *Identity*: there is an element $1 \in G$ such that $1x = x1 = x$ for all $x \in G$.
(3) *Inverse*: if $x \in G$ there is an element $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = 1$.

If further G satisfies

(4) *Commutative law*: if $x, y \in G$ then $xy = yx$.

G is said to be *an abelian group* (after *Abel*).

A non-empty subset H of G is said to be *a subgroup* of G, if $xy^{-1} \in H$ for every $x, y \in H$. We indicate this by $H \leq G$. Note that H itself is a group and H and G has the same identity. If $X \subseteq G$, *the subgroup generated by* X is the intersection $\langle X \rangle$ of all subgroups of G containing X. Actually, if $X^{-1} = \{x^{-1} \mid x \in X\}$, it is not hard to see that

$$\langle X \rangle = \{x_1 \ \cdots \ x_r \mid r \geq 1,\ x_i \in X \cup X^{-1}\}.$$

An *isomorphism* between two groups G_1 and G_2 is a bijective map

$$f : G_1 \longrightarrow G_2$$

such that $f(xy) = f(x)f(y)$ for all $x, y \in G_1$. It is said that G_1 and G_2 are *isomorphic* or *have the same type of isomorphy* and denoted $G_1 \cong G_2$.

Let G be a group. If $x \in G$ and $n \in \mathbb{Z}$, *the* n^{th}*–power* x^n *of* x is

$$x^n = \begin{cases} x \ \cdots \ x \ (n \text{ times}, & \text{if } n > 0 \\ 1, & \text{if } n = 0 \\ x^{-1} \ \cdots \ x^{-1} \ (|n| \text{ times}, & \text{if } n < 0 \end{cases}$$

Clearly, $x^n x^m = x^{n+m}$ and $(x^n)^m = x^{nm}$ for every $x \in G$ and $n, m \in \mathbb{Z}$. The least $n > 0$ such that $x^n = 1$ is called *the order of* x. If such an n does not exist, x is said to have *infinite order.* A group G is said to be *finite* if the underlying set G is a finite set; in this case, the cardinal of G is called *the order of* G and denoted by $|G|$. A non-finite group is called *infinite.* If G is a finite group and $x \in G$ then the order of x is finite and divides $|G|$.

1.2 Examples

(1) $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{Q} \setminus \{0\}, .)$, $(\mathbb{R}, +)$, $(\mathbb{R} \setminus \{0\}, .)$ $(\mathbb{C}, +)$, $(\mathbb{C} \setminus \{0\}, .)$ are examples of infinite abelian groups.

(2) If $n > 1$ is given, *the complex n^{th}–roots of unity* under multiplication, *the integers module n* under sum and *the rotations fixing a regular n-gon* under composition are examples of isomorphic abelian groups of order n. The common type of isomorphy is called *the (finite) cyclic group of order n* and will be denoted by C_n. On the other hand, it can be shown that a finite abelian group is a cartesian product of cyclic groups (see [42])

(3) In general, the plane movements fixing a regular n-gon under composition form a non-abelian group whose type of isomorphy is called *the dihedral group D_n* of order $2n$. This group can be generated by a rotation and an axis symmetry and contains the n rotations fixing the n-gon, that is $C_n \leq D_n$.

(4) A bijection σ of $\{1, \cdots, n\}$ into itself is called *a permutation* on n cyphers. If $n \geq 3$, the permutations on n cyphers under composition of maps compose a finite non-abelian group of order $n!$ called *the symmetric group* of degree n and denoted S_n. Clearly, $D_n \leq S_n$. The n–tuple $(\alpha(1), \cdots \alpha(n))$ contains a number of inversions whose parity is an invariant ruled out by the rule of signs. If follows that the even permutations form a subgroup A_n of the symmetric group called *the alternating group* of degree n. $|A_n| = n!/2$.

(5) Let K be a field and let $n \geq 2$. The invertible matrices

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

whose entries are all in K form a group under multiplication called *the general linear group* on K of dimension n and denoted by $GL(n, K)$. This group can also be thought as the group of all invertible linear operators of a vector space V over K of dimension n and in this case is denoted by $GL(V)$.

If K is a finite field then $GL(n, K)$ is finite. If $|K| = q = p^r$ (p a prime), we denote it by $GL(n, q)$ instead of $GL(n, K)$. We have

$$|GL(n, q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}).$$

A subgroup $G \leq GL(K)$ is usually known as *a linear group.* For example, *the special linear group $SL(n, K)$* composed for all matrices whose determinant is 1 is a linear group.

This group is *the commutator subgroup* of $GL(n, K)$ as it can be generated by the elements of the form $x^{-1}y^{-1}xy,\ x, y \in GL(n, K)$. This remark will be very important in the constructions of classical groups.

On the other hand, it can be shown that *any finite group is linear* as the so-called regular representation or Cayley representation shows ([42]).

1.3 Normal Subgroups and Simple Groups

A subgroup H of a group G defines an equivalence relation on G by

$$x \sim_l y \Longleftrightarrow x^{-1}y \in H.$$

The equivalence class of x is the subset $xH = \{xh \mid h \in H\}$, called *the left coset* of x module H. Similarly,

$$x \sim_r y \Longleftrightarrow yx^{-1} \in H$$

is another equivalent relation and Hx is *the right coset* of an element $x \in G$ module H. Obviously left and right cosets of a group module a subgroup can be different. However, if G is a finite group, since $|xH| = |H| = |Hx|$, the number of left and right cosets coincide. This number is called *the index* of H in G and denoted by $|G : H|$. We have $|G| = |H||G : H|$ (theorem of *Lagrange*).

A subgroup N of a group G is said to be *normal* if $xN = Nx$ for all $x \in G$ (equivalently, if N is G*–invariant*, that is $x^{-1}Nx = N$ for every $x \in G$) and will be denoted by $N \trianglelefteq G$. For example, $C_n \trianglelefteq D_n$ and $A_n \trianglelefteq S_n$. In an arbitrary group G, the trivial subgroup $\langle 1 \rangle$ and G itself are always normal subgroups of G. Actually G is said to be a *simple group* if $\langle 1 \rangle$ and G are the only normal subgroups of G. The cyclic group C_p of prime order p is simple as well as the alternating group A_n for $n \geq 5$.

If N is a normal subgroup of a group G then the underlying quotient set consisting of the cosets of G module Hcan be endowed with a law of group given by

$$(Nx)(Ny) = Nxy.$$

This group is called *the quotient group* of G by N and denoted by G/N. If G is finite, by Lagrange's theorem, $|G/N| = |G|/|N|$.

2 The Classification of the Finite Simple Groups

Throughout this section, *group will mean finite group.*

As we mentioned above, a fair consequence of Lagrange's theorem establishes that a cyclic group of prime order is simple. In a certain sense, this fact was probably managed by *Klein*. On the other hand, the simplicity of the alternating groups is closely linked to the unsolvability by radicals of the algebraic equations of degree $n \geq 5$ (see [46]). Other simple groups, as *the Mathieu groups* (the first five *sporadic* groups), were constructed within the XIX Century (see [33, 34, 35]) and other simple groups appeared at first in an isolated way. In fact, there is no a precise date in which the Classification of the Finite Simple Groups (shortly, the CFSG) began. Despite this, some highlighting facts happened (especially around 1955-1958) and they stimulated very much the classification. It is worth to mentioning that its proof is not the usual proof of several theorems as it runs to more or less between 10000 and 15000 journal pages, spread across some 500 separate articles by more than 100 mathematicians, almost all written between 1950 and the early 1980's. Moreover, it was not until the 1970's that a global strategy was developed for attacking the complete classification problem, while, in addition, new simple groups were being discovered especially the twenty-one remainder *sporadic* groups. The full theorem was not even possible in precise form until 1980. See [22, 36] for more details.

From the above comments, it is easily understood that the complexity of that proof will obey to concern ourselves to some facts around the CSFG, which allow to have a more precise idea about it.

2.1 Composition Factors

Let G be a group. *A composition series of* G is a chain of subgroups of G

$$\langle 1\rangle = G_0 \trianglelefteq G_1 \trianglelefteq \cdots G_i \trianglelefteq G_{i+1} \trianglelefteq \cdots G_n = G$$

such that, for every $0 \leq i \leq n-1$, G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is a simple group. The set

$$\{G_n/G_{n-1}, \cdots, G_{i+1}/G_i, \cdots, G_1/G_0\}$$

are called *the composition factors of the series.* Since G has finitely many subgroups only, it is immediate that G possesses composition series. We have

Theorem 2.1.1 *(Jordan-Hölder, see [42, Theorem 7.9]) Two composition series of a group have isomorphic composition factors.*

Thus, every group G determines a finite family of simple groups, which say a lot about the structure of the group. The work of *Galois* in characterizing the solvability by radicals of algebraic equations by means of the group of them (see [46]) allowed to introduce the concept of *soluble* (or *solvable*) *group*, which can be characterized by means of composition factors. Actually a group G is soluble if and only if the composition factors of G are cyclic of prime order. It follows that a simple group G is soluble if and only if is abelian, what happens if and only if G is cyclic of prime order. Apparently solvable groups are placed opposite of non-abelian simple groups and so are the problems considered between these two type of groups. Note that there exist non-solvable groups that are not simple, for example the symmetric groups S_n, $n \geq 5$.

Another important consequence of the theorem of Jordan-Hölder is worth to be mentioned. At a first sight one could think that the knowledge of all simple groups would describe all groups in a similar way that a natural number is described as the product of finitely many primes. Unfortunately, the analogy is false as non-isomorphic groups can have equal composition factors. For example, the cyclic group of order 4 C_4 and the 4–group of *Klein* $C_2 \times C_2$ are not isomorphic but have $\{C_2, C_2\}$ as composition factors. However, the question of the knowledge of simple groups and the CFSG appear to be very natural.

2.2 Towards the Classification

As we mentioned above, the cyclic groups of prime order are the only abelian simple groups. In 1955, Burnside raised the following question.

Conjecture *There are no non-abelian simple groups of odd order.*

This conjecture is shown to be true by the celebrated theorem of Feit-Thompson [17], one of the achievements of this theory, which runs over about 255 pages of the Pacific Journal in Mathematis.

Theorem 2.2.1 *A group of odd order is soluble.*

Corollary 2.2.2 *A non-abelian simple group has even order divisible by* 4.

Previously to this, Richard Brauer (Berlin 1901 – Velmont MA 1977), one of the fathers of the CFSG had proposed an inductive treatment of the problem, of which the theorem of Feit-Thompson is probably the first success. With the aid of his student K.A. Fowler, Brauer had characterized the simplicity of some linear groups, but the fundamental contribution of Brauer and Fowler to this problem can be deduced to their fundamental work on groups of even order ([10]). Let G be a group. *An involution of* G is an element $i \in G$ of order 2. An elemental result of the Theory of Groups asserts that every group of even order has involutions (see [42]). On the other hand, if $x \in G$, *the centralizer of x in G* is

$$C_G(x) = \{g \in G \mid gx = xg\}.$$

Now, we can state the theorem of Brauer-Fowler [10].

Theorem 2.2.3 *Let G be a simple group of even order, and let $i \in G$ be an involution. Then $C_G(i) \neq G$, and if $|C_G(i)| = m$ then $|G| \leq (\frac{1}{2}m(m+1))!$.*

This result raises the possibility of characterizing simple groups G in terms of the structure of the centralizer of an involution of G, which is a group of smaller order than G. In fact, the following consequence is immediate.

Corollary 2.2.4 *Let H be a group of even order with an involution $j \in Z(H) = C_H(H)$. Then there are at most finitely many types of simple groups G having an involution i such that $C_G(i) \cong H$.*

As a consequence, Brauer develop a procedure to classify simple groups, which is introduced to the mathematical community with the occasion of the International Congress of Mathematics held at Amsterdam in 1954.

Method of centralizers of involutions

Start with a known non-abelian finite simple group S and an involution $u \in S$, and let $K = C_S(u)$. Consider now simple non-abelian groups G having an involution i such that $C_G(i) \cong S$. There are only finitely many types of such groups G, one of which is S itself by construction. Actually, if there is only one type, we have $G \cong S$, and then a characterization theorem for S has been established in terms of the structure of the centralizer of one of its involutions, a group of smaller order than S. If the attempt fails because there are groups not isomorphic to S among the groups G, there may be previously unknown simple groups among the groups G.

This procedure has been a source of discovery of several new finite simple groups. More details in [18, 19, 20, 42, 43].

2.2.1 Classical Groups and Groups of Lie Type

The so-called *classical groups* are (finite) groups belonging to three big families, namely

- *Linear groups*
- *Symplectic and Orthogonal groups*
- *Unitary groups*

We have already defined the general linear group $GL(n, q)$ and the special linear group $SL(n, q)$, which is its commutator subgroup. By definition, *the projective special linear group* $L(n, q) = PSL(n, q) = SL(n, q)/Z$ is the quotient group of the special linear group by the normal subgroup formed by their scalar matrices. With the exception of lower cases ($n = 2$ and $q = 2, 3$), $L(n, q)$ is know to be simple. *The symplectic and orthogonal groups* can be defined in a similar way starting of subgroups of $GL(n, q)$ consisting of the matrices leaving invariant a given non-degenerate alternating bilinear form or a quadratic form, respectively, on the underlying n–dimensional vector space on which $GL(n, q)$ naturally acts. Then we consider the corresponding commutator subgroup and the quotient groups of these by their scalar matrices. This procedure leads to families of groups $PSp(n, q)$ and $P\Omega^{\pm}(n, q)$, most of which are simple groups. *The unitary groups* $U(n, q) = PSU(n, q)$ are constructed following a similar procedure; in this case, the starting group is the subgroup of $GL(n, q)$ consisting of the matrices that are invariant by the automorphism α of $GL(n, q)$ given by

$$\alpha(M) = ((\overline{M})^t)^{-1}.$$

The classical groups are known to be analogues of the complex Lie groups A_n, B_n, C_n and D_n (the interested reader can see [15, 51] for details on this and related topics). Although finite analogues of the exceptional Lie groups were constructed by *Dickson* by the early part of the XXthcentury, it was not until 1955 that Chevalley [12] showed by a general Lie-theoretic argument that there exist finite analogues $\mathcal{L}(q)$ of every semisimple complex Lie group $\mathcal{L}(q)$ associated to a finite field with q elements. In particular, he proved the existence of the five families $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$ of exceptional simple groups of Lie type. At the same time, Tits [47] was giving geometric constructions of several of these famu1es. These groups $\mathcal{L}(q)$, with $\mathcal{L} = A_n,\ B_n,\ C_n,\ D_n,\ G_2,\ F_4,\ E_6,\ E_7$ and E_s, are called *the untwisted groups of Lie type* or *the Chevalley groups*. The more technical construction of *the twisted groups* followed soon after (see details in the book by Steinberg [44]).

2.2.2 The Sporadic Groups

The sporadic groups are the 26 simple groups that do not fit into any of the four infinite families of simple groups we have just described (i.e., the cyclic groups of prime order, alternating groups of degree at least five, twisted and untwisted Lie-type groups). They have been constructed in several ways, usually as the automorphisms of a geometric configuration. The smallest sporadic group is the Mathieu group, which has order 7920, and the largest is *the monster group*, which has order 808017424794512875886459904961710757005754368000000 The orders of the sporadic groups given in increasing order are 7920, 95040, 175560, 443520, 604800, 10200960, 44352000, 50232960, ... A summary of sporadic groups can be found in [1, 15, 50, 51]. We only recall their names.

1. *The Mathieu groups*: $M_{11},\ M_{12},\ M_{22},\ M_{23},\ M_{24}$
2. *The Janko groups*: $J_1,\ J_2,\ J_3,\ J_4$
3. *The Conway groups*: $Co_1,\ Co_2,\ Co_3$
4. *The Fischer groups*: $Fi_{22},\ Fi_{23},\ Fi_{24}$
5. *Other groups*: $HS,\ McL,\ He,\ Ru,\ Suz,\ O'N,\ Th,\ Ly,\ HN$

Higman-Sims, McLaughlin, Held, Rudvalis, O'Nan, Suzuki, Thompson, Lyons, Harada-Norton, respectively.

6. *The monster of Fischer-Gries*: M

7. *The baby monster*: F

2.3 The Theorem of Classification

We state the theorem of the classification.

Theorem 2.3.1 *Let G be a finite simple group. Then G is isomorphic to one group of the following*

(1) A cyclic group of prime order;

(2) An alternating group of degree at least five;

(3) A twisted Lie-type group;

(4) An untwisted Lie-type group; or

(5) A sporadic group.

3 After the Classification of the Finite Simple Groups

The classification of the finite simple groups was an ongoing organic process, whose progress in the last decades of the XXth Century since the Odd Order Theorem (the theorem of Feit-Thompson) was extraordinary. Many significant questions and conjectures wee suddenly accessible thanks to the completion of the Classification and its consequences. Surveys on some of this work are available in [19, 20, 36]. Moreover new results were proved by checking the CFSG. It would be impossible to tell in short the numerous progress made.

On the other hand, quite a bit of recent research in finite group theory has developed in response to problems from other areas of mathematics. Here we briefly mention some of the active areas of research (see [43]).

- *Representation Theory.*
- *Maximal Subgroups and Primitive Permutation Representations.*
- *Theory of Graphs.*
- *Field Theory.*
- *Geometry and Topology.*
- *Model Theory.*

Nowadays, within the Theory of Finite Groups, there is not an objective having the importance that the CFSG had. The group-theoretical questions that are now object of study deals with several aspects of *composite groups*.

3.0.1 The Revision Project

The process of *revision* of the classification was for years inextricably associated with the name of Helmut Bender, who began the creative revision of the Odd Order Theorem and successive attempts to explain other different pieces of this research are due to Glauberman, Enguerhard, Goldschmidt, Aschbacher and others. In 1982, Gorenstein, who together with Brauer took an overview of the whole project and steered it to a successful conclusion, launched a *revision project* in which he was joined by Lyons and Solomon. This project is intended to complement the work of the other revision efforts to yield a new and complete proof of the Classification. To realize this, they conceive a project of a dozen books, *the GLS series* (of which five have already appeared: [22, 23, 24, 25, 26]), which Gorenstein was not be able to see a cause of his death in 1992. We simply present here the statement of the Revised Theorem. The status of the project and its interfaces with other revision efforts are available in the GLS series. See also [2] to learn more on the status of the revision project.

A $\mathcal{K}$–group is a finite group H such that if S is a simple quotient of a subgroup of H then S is isomorphic to one of the simple groups listed in Theorem 2.3.1. Here the letter "$\mathcal{K}$" is used as an abbreviation for "known". A group G is said to be *$\mathcal{K}$–proper* if every proper subgroup of G is a $\mathcal{K}$–group. With this terminology, the statement of the Classification Theorem is

Theorem 3.0.2 *If G is a $\mathcal{K}$–proper finite simple group then G is a $\mathcal{K}$–group.*

The proof is an application of the technique known as *minimal counterexample*, that is it is supposed that the result is false and it is chosen a $\mathcal{K}$–proper finite simple group G that is not a $\mathcal{K}$–group with least order in order to arrive to a contradiction.

Clearly the term "$\mathcal{K}$–group" is only used as part of the proof of Theorem 3.0.2. Indeed, that theorem states that *every finite simple group is isomorphic to a known simple group* and it follows that *every finite group is a $\mathcal{K}$–group.*

3.0.2 Monsterology

The monster group M is the most popular among the sporadic groups. It was conjectured in 1972 by Fischer in 1972 and constructed by Griess in 1982; details on its uniqueness were checked by Norton around 1983. It is a group of rotations of a complex space of dimension 196883 whose order ($\simeq 10^{54}$) is bigger than the number of elementary particles of Jupiter!

It is worth to mentioning that many of the properties of the monster group have been discovered before it was constructed. The monster remains the single most tantalizing simple group, with apparent (but as yet mysterious) connections to Kac-Moody Lie theory, quantum field theory, modular functions, and congruence subgroups of $SL(2, \mathbb{Z})$. It is well known how to show that a finite group generated by involutions acts analytically on a Riemann surface. It would be interesting to understand higher-dimensional analytic representations of finite (simple) groups, which would clarify the connections between the Monster (and its subgroups) and classical elliptic modular functions.

A concrete relationship between these ideas was carry out in the work of Borcherds. Richard E. Borcherds is a professor of Mathematics at Berkeley CA, which was awarded by a 1999 Fields Medal by his proof of a conjecture of Conway and Norton ([13, 14, 16]), which establishes a connection between the dimensions of the irreducible representations of the monster group and certain modular functions of weight 0 ([4, 5, 6, 7]). To do this, he invented *the vertex algebras* a generalization of commutative rings with derivation, which are closely linked to elements of *the String Theory* ([5]). The connection produces the so-called *Moonshine* that are a very popular object of study (see [3, 8, 13]).

3.0.3 Infinite Groups

Would it be possible to classify infinite simple groups? The general answer to this question is no because the existence of the so called *Tarski's Monsters*, which are infinite groups with all proper subgroups finite. These groups were constructed by Ol'shanskii [39].

Theorem 3.0.3 *For every odd prime p, there exists an infinite group G whose proper subgroups are cyclic of order p.*

Clearly, this G is simple, can be generated by two elements and all non-trivial elements of G have order p. The groups of Ol'shanskii disprove many conjectures, as the general problem of Burnside, avoid the classification of infinite groups under several criteria, for example, simplicity, and indicate that *a finiteness condition* has to be imposed in classification's problems of infinite groups to avoid their presence.

However, something can be done in the frame of infinite groups and we will give a brief notice about (see also [40]). A group G is said to be *locally finite* if finitely generated subgroups of G are finite. The question of classifying simple infinite locally finite groups was initiated in 1964 by Kegel [28]. Examples, constructions and properties of locally finite groups are available in [29] and also in [41]. In fact, it is very easy to construct infinite locally finite simple groups. For example, for every $n \geq 1$, we embed the symmetric group S_n in the symmetric group S_{n+1} simply adding the cypher $n+1$ in such a way that the latter is fixed and the others move as in the lower symmetric group. This embedding keeps the parity of the number of inversions so that it induces an inclusion $A_n \longrightarrow A_{n+1}$. Transforming the embeddings in inclusions, we have an infinite chain of subgroups

$$A_1 \leq A_2 \leq \cdots A_5 \leq \cdots A_n \leq A_{n+1} \leq \cdots$$

whose union is an infinite locally finite simple group (called *the restricted infinite alternating group*). We refer to [29] to find more details.

It is rather easy to see that the question can be reduced to the study of countable groups. Indeed, *a group G is simple if and only if it has a local system of subgroups* (i.e. G is the union of an upper directed set of subgroups) *that are countable and simple.* The structure of countable locally finite simple groups is

Theorem 3.0.4 *Let G be a countably infinite locally finite simple group. Then there exists an ascending chain of subgroups of G*

$$G_1 \leq G_2 \leq \cdots \leq G_n \leq G_{n+1} \cdots \leq$$

such that the following conditions are satisfied:

(1) For each $n \geq 1$, G_n contains a maximal normal subgroup M_n such that $G_n \cap M_{n+1} = \langle 1 \rangle$; and

(2) $G = \bigcup_{n \geq 1} G_n$.

Independently, Borovik [9], Hartley and Shute [27] and Thomas [48, 49] showed a fundamental result that can be enounced as follows.

Theorem 3.0.5 *Suppose that G is a group having a countable local system of finite subgroups $\{G_n\}$ such that there exists a Chevalley functor $C(d, --)$ satisfying $G_n = C(d, F_n)$, where F_n is a finite field. Then there exists a locally finite field F such that $G \cong C(d, F)$.*

This essentially asserts that the countably infinite locally finite simple groups are the infinite locally finite versions of the types described in Theorem 2.3.1, that is the locally finite versions of cases (2)-(3)-(4).

A fairly consequence is the following result.

Corollary 3.0.6 *An infinite simple periodic linear group is a group of type Lie-Chevalley over a locally finite field.*

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